موسوعة التكامل

IntegrationPedia

Prepared by- إعداد

أساتذة منتديات الرياضيات العربية

Members of Arabic Math Forums

typing and design- طباعة

Jud El Harf - الأستاذة جود الحرف

uaemath

Part One – الجزء الأول

http://www.uaemath.com/ar/aforum

Set (1)

Solved by

Solvers	Integral
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق tinaamina1990 - hidaya - ali2008	1
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق tinaamina1990 - hidaya - ali2008	2
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق tinaamina1990 - ali2008	3
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق tinaamina1990 - ali2008	4
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق ali2008	5
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق - صادق العلي ali2008	6
عبد العاطي وفا - عبد الواحد - فريج عبد الرزاق - صادق العلي ali2008	7
عبد الواحد - فريج عبد الرزاق - صادق العلي ali2008	8
عبد الواحد - فريج عبد الرزاق - صادق العلي ali2008	9
عبد الواحد - فريج عبد الرزاق - صادق العلي ali2008	10

Set(1) - المجموعة الأولى

جدول الدوال غير المحدودة- Indefinite Integrals

1- الدوال التي تحتوي على (a+ b x) وتكون درجة هذه العبارة عدداً صحيحاً

1) $\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + c$

2) $(a+bx)^n dx = \frac{(a+bx)^{n+1}}{b(n+1)} + c \, , n \neq -1$

3) $\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}\left(a+bx - a\ln(a+bx)\right) + c$

4) $\int \frac{x^2 dx}{a+bx} = \frac{1}{b^3}\left(\frac{1}{2}(a+bx)^2 - 2a(a+bx) + a^2\ln(a+bx)\right) + c$

5) $\int \frac{dx}{x(a+bx)} = -\frac{1}{a} ln\frac{a+bx}{x} + c$

6) $\int \frac{dx}{x^2(a+bx)^2} = -\frac{1}{ax} + \frac{b}{a^2} ln\frac{a+bx}{x} + c$

7) $\frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left(\ln(a+bx) + \frac{a}{a+bx}\right) + c$

8) $\int \frac{x^2 dx}{(a+bx)^2} = \frac{1}{b^3}\left(a+bx - 2a\ln(a+bx) - \frac{a^2}{a+bx}\right) + c$

9) $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\frac{a+bx}{x} + c$

10) $\int \frac{x\,dx}{(a+bx)^2} = \frac{1}{b^2}\left(-\frac{1}{a+bx} + \frac{a}{2(a+bx)^2}\right) + c$

1) $I = \int \frac{dx}{a+bx} = \frac{1}{b}\int \frac{b}{a+bx}dx$ و $I = \frac{1}{b}\ln|a+bx| + c$

Example

$$\int \frac{dx}{2+3x} = \frac{1}{3}\ln|2+3x| + c$$

2) $\mathrm{I} = \int (a+bx)^n\, dx = \frac{1}{b}\int b\,(a+bx)^n\ dx$ و $I = \frac{(a+bx)^{n+1}}{b(n+1)} + c$

Example

$$\int (2+3x)^5\, dx = \frac{(2+3x)^6}{18} + c$$

3) $\mathrm{I} = \int \frac{x}{a+bx}dx = \frac{1}{b}\int \frac{bx}{a+bx}dx = \frac{1}{b}\int \frac{a+bx-a}{a+bx}dx = \frac{1}{b}\int \left(1-\frac{a}{a+bx}\right)dx$

$= \frac{1}{b}\int \left(1-\frac{a}{a+bx}\right)dx$ و $I = \frac{1}{b}\left(x-\frac{a}{b}\ln(a+bx)\right) + c$

Example

$$\int \frac{x}{5+3x}dx = \frac{1}{3}(x-\frac{5}{3}\ln(5+3x)) + \mathrm{c}$$

4) $\mathrm{I} = \int \frac{x^2}{a+bx}dx = \int \left(\frac{x}{b}-\frac{a}{b^2}+\frac{\frac{a^2}{b^2}}{ax+b}\right)dx$ و $I = \frac{x^2}{2b} - \frac{a}{b^2}x + \frac{a^2}{b^3}\ln|a+bx| + c$

Example

$$\int \frac{x^2}{1+4x}dx = \frac{x^2}{8} - \frac{1}{16}x + \frac{1}{64}\ln|\ 1+4x| + c$$

5) $I = \int \frac{dx}{x(a+bx)}$

let : $y = \frac{a+bx}{x} = \frac{a}{x} + b$

$dy = \frac{-a}{x^2}\ dx$

$\int \frac{dx}{x(a+bx)} = \frac{-1}{a}\int \frac{\mathrm{dy}}{\mathrm{y}} = \frac{-1}{\mathrm{a}}\ln|\mathrm{y}| + \mathrm{c}$ و $I = \frac{-1}{\mathrm{a}}\ln\left|\frac{ax+b}{x}\right| + c$

Example

$$\int \frac{dx}{x(5+2x)} = \frac{-1}{5}\ln\left|\frac{5x+2}{x}\right| + c$$

6) $I=\int \frac{1}{x^2(a+bx)}dx=\int \frac{(a+bx)-bx}{a\cdot x^2(a+bx)}dx$

$=\int [\frac{(a+bx)}{a\cdot x^2(a+bx)}+\frac{-b}{a\cdot x(a+bx)}]dx=\underbrace{\int [\frac{1}{a\cdot x^2}dx}_{I1}-\frac{b}{a^2}\underbrace{\int \frac{a}{x(a+bx)}]dx}_{I2}$

$I1=\int \frac{1}{a\cdot x^2}dx=\frac{-1}{a\cdot x}+C1$

$I2=\int \frac{a}{x(a+bx)}dx=\int \frac{(a+bx)-bx}{x(a+bx)}dx=$

$\int [\frac{1}{x}-\frac{b}{a+bx}]dx=\ln|x|-\ln|a+bx|+C2$

$I=I1+I2$

$\Rightarrow \boxed{I=\frac{-1}{a\cdot x}+\ln|x|-\ln|a+bx|+(C1+C2)}$

Example

$$\int \frac{dx}{x^2(3+2x)}=\frac{-1}{3x}+\ln|x|-ln|3+2x|+c$$

7) $I=\int \frac{x}{(a+bx)^2}dx$

let : $y=a+bx\Rightarrow x=\frac{y-a}{b}\Rightarrow dx=\frac{dy}{b}$

$\Rightarrow I=\int \frac{\frac{y-a}{b}}{(y)^2}\cdot\frac{dy}{b}=\int \frac{y-a}{(y)^2}dy=\int [\frac{1}{y}-\frac{a}{y^2}]dy=\ln|y|+\frac{a}{y}+C$

$\Rightarrow \boxed{I=\ln|a+bx|+\frac{a}{a+bx}+C}$

Example

$$\int \frac{x}{(2+3x)^2}dx=\ln|2+3x|+\frac{2}{2+3x}+c$$

8) $I=\int \frac{x^2}{(a+bx)^2}dx$

let : $y=a+bx\Rightarrow x=\frac{y-a}{b}\Rightarrow dx=\frac{dy}{b}$

$\Rightarrow I=\int \frac{(\frac{y-a}{b})^2}{y^2}\cdot\frac{dy}{b}=\frac{1}{b^3}\cdot\int \frac{y^2-2ay+a^2}{y^2}dy=\frac{1}{b^3}\cdot\int [1-\frac{2a}{y}+\frac{a^2}{y^2}]dy$

$=\frac{1}{b^3}[y-2a\ln|y|-\frac{a^2}{y}]+C$

$\Rightarrow \boxed{I=\frac{1}{b^3}[a+bx-2a\ln|a+bx|-\frac{a^2}{a+bx}]+C}$

Example

$$\int \frac{x^2}{(5+2x)^2}dx=\frac{1}{8}\left(5+2x-10\ln|5+2x|-\frac{25}{5+2x}\right)+c$$

9) $I=\int\frac{1}{x(a+bx)^2}dx=\int\frac{(a+bx)-bx}{a\cdot x(a+bx)^2}dx=\int[\frac{(a+bx)}{a\cdot x(a+bx)^2}+\frac{-bx}{a\cdot x(a+bx)^2}]dx$

$$=\int[\underbrace{\frac{1}{a\cdot x(a+bx)}}_{I1}+\underbrace{\frac{-b}{a\cdot(a+bx)^2}}_{I2}]dx$$

$$I1=\int\frac{1}{a\cdot x(a+bx)}dx=-\frac{1}{a^2}\cdot\int\frac{bx-a-bx}{x(a+bx)}dx=-\frac{1}{a^2}\cdot\int[\frac{b}{a+bx}-\frac{1}{x}]dx$$

$$=-\frac{1}{a^2}\cdot[\ln|a+bx|-\ln|x|]+C1=-\frac{1}{a^2}\cdot\ln|\frac{a+bx}{x}|+C1$$

$$I2=\int\frac{-b}{a\cdot(a+bx)^2}dx=\frac{1}{a}\cdot\int\frac{-b}{(a+bx)^2}dx=\frac{1}{a(a+bx)}+C2$$

$$I=I1+I2$$

$$\Rightarrow \boxed{I=-\frac{1}{a^2}\cdot\ln|\frac{a+bx}{x}|+\frac{1}{a(a+bx)}+(C1+C2)}$$

Example

$$\int\frac{dx}{x(5+2x)^2}=\frac{-1}{25}\ln\left|\frac{5+2x}{x}\right|+\frac{1}{5(5+2x)}+c$$

10) $I=\int\frac{x}{(a+bx)^3}dx$

$$let: a+bx=y\Rightarrow dx=\frac{dy}{b}\Rightarrow I=\int\frac{(\frac{y-a}{b})}{y^3}\cdot\frac{dy}{b}=\frac{1}{b^2}\cdot\int\frac{y-a}{y^3}dy$$

$$=\frac{1}{b^2}\cdot\int[\frac{1}{y^2}-\frac{a}{y^3}]dy=\frac{1}{b^2}\cdot[-\frac{1}{y}+\frac{a}{2y^2}]+C$$

$$\Rightarrow \boxed{I=\frac{1}{b^2}\cdot[-\frac{1}{a+bx}+\frac{a}{2(a+bx)^2}]+C}$$

Example

$$\int\frac{x}{(7+2x)^3}=\frac{1}{4}\left(-\frac{1}{(7+2x)}+\frac{7}{2(7+2x)^2}\right)+c$$

Set(2) - المجموعة الثانية

الدوال التي تحتوي على $a^2 + x^2, a^2 - x^2, a + bx^2$

11) $\int \frac{dx}{1+x^2} = \arctan x + c$

12) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} arctan \frac{x}{a} + c$

13) $\frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + c$

14) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{x+a}{x-a} + c$

15) $\int \frac{dx}{a + bx\%2} = \frac{1}{\sqrt{ab}} \arctan x \sqrt{\frac{b}{a}} + c \ (a > 0 , b > 0)$

16) $\int \frac{\mathrm{dx}}{\mathrm{a} - \mathrm{bx}^2} = \frac{1}{2\sqrt{\mathrm{ab}}} \ln \frac{\sqrt{\mathrm{a}} - \mathrm{x}\sqrt{\mathrm{b}}}{\sqrt{\mathrm{a}} - \mathrm{x}\sqrt{\mathrm{b}}}$

17) $\int \frac{xdx}{a+bx^2} = \frac{1}{2b} \ln\left(x^2 + \frac{a}{b}\right) + c$

18) $\int \frac{x^2dx}{a+bx} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a+bx}$

19) $\int \frac{dx}{x(a+bx^2)} = \frac{1}{2b} ln \frac{x^2}{a+bx^2} + c$

20) $\int \frac{dx}{x^2(a + bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{dx}{a+bx^2}$

21) $\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a\ (a+bx^2)} + \frac{1}{2a} \int \frac{dx}{a+bx^2}$

Set (2)

Solved by

Solvers	Integral
صادق العلي - عبد الواحد	11
صادق العلي - عبد الواحد	12
صادق العلي - عبد الواحد	13
صادق العلي	14
صادق العلي - عبد الواحد	15
صادق العلي - عبد الواحد	16
صادق العلي - عبد الواحد	17
صادق العلي - عبد الواحد mourad24000	18
صادق العلي - عبد الواحد mourad24000	19
hamza_mn	20
laila245	21

11) $I = \int \frac{dx}{1+x^2}$

let : $x = \tan y \Rightarrow dx = \sec^2(y)dy$

$$\Rightarrow I = \int \frac{\sec^2(y)dy}{1+\tan^2(y)} = \int \frac{\sec^2(y)dy}{\sec^2(y)} = \int dy = y + C$$

$$\Rightarrow \boxed{I = \tan^{-1}(x) + C}$$

Example

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$$

12) $I = \int \frac{dx}{a^2+x^2}$

let : $x = a \cdot \tan y \Rightarrow dx = a \cdot \sec^2(y)dy$

$$\Rightarrow I = \int \frac{a \cdot \sec^2(y)dy}{a^2 + a^2 \cdot \tan^2(y)} = \int \frac{a \cdot \sec^2(y)}{a^2(1+\tan^2(y))}dy$$

$$= \frac{1}{a} \cdot \int \frac{\sec^2(y)}{(1+\tan^2(y))}dy = \frac{1}{a} \cdot \int \frac{\sec^2(y)}{\sec^2(y)}dy = \frac{1}{a} \cdot \int dy = \frac{1}{a} \cdot y + C$$

$$\Rightarrow \boxed{I = \frac{1}{a} \cdot \tan^{-1}(\frac{x}{a}) + C}$$

Example

$$\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}(\frac{x}{2}) + c$$

13) $I = \int \frac{dx}{a^2-x^2} = \int \frac{1}{(a-x)(a+x)}dx = \int \frac{1}{(a-x)(a+x)}dx$

$$\frac{1}{(a-x)(a+x)} = \frac{M}{(a-x)} + \frac{N}{(a+x)} = \frac{M(a+x) + N(a-x)}{(a-x)(a+x)}$$

$$= \frac{a(M+N) + x(M-N)}{(a-x)(a+x)}$$

$$\Rightarrow M + N = \frac{1}{a} \quad ,,, \quad M - N = 0$$

$$\Rightarrow M = N = \frac{1}{2 \cdot a}$$

$$\Rightarrow I = \int [\frac{\frac{1}{2 \cdot a}}{(a-x)} + \frac{\frac{1}{2 \cdot a}}{(a+x)}]dx = \frac{-1}{2 \cdot a} \cdot \ln|a-x| + \frac{1}{2 \cdot a} \cdot \ln|a+x| + C$$

$$= \boxed{\frac{1}{2 \cdot a} \cdot \ln|\frac{a+x}{a-x}| + C}$$

Another solution: $I = \int \frac{dx}{a^2-x^2} = \frac{1}{2a} \cdot \int \frac{2a}{a^2-x^2}dx = \frac{1}{2a} \cdot \int \frac{(a+x)+(a-x)}{(a+x)(a-x)}dx$

$$= \frac{1}{2a} \cdot \int [\frac{1}{a+x} + \frac{1}{a-x}]dx = \frac{1}{2a} \cdot [\ln|a+x| - \ln|a-x|] + C$$

$$\Rightarrow \boxed{I = \frac{1}{2a} \cdot \ln|\frac{a+x}{a-x}| + C}$$

Example

$$\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + c$$

14) $I = \int \frac{dx}{a^2 - x^2} = \int \frac{-1}{x^2 - a^2}dx = \frac{1}{2\cdot a}\cdot\int\frac{-2\cdot a}{x^2-a^2}dx$

$= \frac{1}{2\cdot a}\cdot\int\frac{(x-a)-(x+a)}{(x-a)(x+a)}dx = \frac{1}{2\cdot a}\cdot\int[\frac{1}{x+a}-\frac{1}{x-a}]dx$

$= \frac{1}{2\cdot a}\cdot[\ln|x+a| - \ln|x-a|]+C$

$\Rightarrow \boxed{I = \frac{1}{2\cdot a}\cdot\ln|\frac{x+a}{x-a}|+C}$

Example

$$\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + c$$

15) $I = \int \frac{dx}{a+bx^2}$

$let: x = \sqrt{\frac{a}{b}}\cdot\tan(y) \Rightarrow dx = \sqrt{\frac{a}{b}}\cdot\sec^2(y)dy \Rightarrow I = \int\frac{\sqrt{\frac{a}{b}}\cdot\sec^2(y)dy}{a+b[\sqrt{\frac{a}{b}}\cdot\tan(y)]^2}$

$= \sqrt{\frac{a}{b}}\cdot\int\frac{\sec^2(y)}{a+a\cdot\tan^2(y)}dy = \sqrt{\frac{a}{b}}\cdot\int\frac{\sec^2(y)}{a\cdot\sec^2(y)}dy = \sqrt{\frac{a}{b}}\cdot\frac{1}{a}\int dy$

$= \frac{1}{\sqrt{ab}}\cdot y + C \Rightarrow \boxed{I = \frac{1}{\sqrt{ab}}\cdot\tan^{-1}(\sqrt{\frac{b}{a}}\cdot x)+C}$

Example

$$\int \frac{dx}{7+5x^2} = \frac{1}{\sqrt{35}}\tan^{-1}(\sqrt{\frac{5}{7}}\ x) + c$$

16) $I = \int\frac{1}{a-bx^2}dx = \frac{1}{2\cdot\sqrt{ab}}\cdot\int\frac{2\cdot\sqrt{ab}}{a-bx^2}dx = \frac{1}{2\cdot\sqrt{ab}}\cdot\int\frac{(\sqrt{ab}+bx)+(\sqrt{ab}-bx)}{(\sqrt{a}+\sqrt{b}\cdot x)(\sqrt{a}-\sqrt{b}\cdot x)}dx$

$= \frac{1}{2\cdot\sqrt{ab}}\cdot\int[\frac{1}{\sqrt{a}+\sqrt{b}\cdot x}+\frac{1}{\sqrt{a}-\sqrt{b}\cdot x}]dx$

$= \frac{1}{2\cdot\sqrt{ab}}\cdot[\ln|\sqrt{a}+\sqrt{b}\cdot x| - \ln|\sqrt{a}-\sqrt{b}\cdot x|]+C$

$\Rightarrow \boxed{I = \frac{1}{2\cdot\sqrt{ab}}\cdot\ln|\frac{\sqrt{a}+\sqrt{b}\cdot x}{\sqrt{a}-\sqrt{b}\cdot x}|+C}$

Example

$$\int \frac{dx}{9-4x^2} = \frac{1}{2\sqrt{32}}\ln\left|\frac{3+2x}{3-2x}\right| + c$$

17) $I = \int \frac{x}{a+bx^2}dx = \frac{1}{2b}\cdot\int\frac{2b\cdot x}{a+bx^2}dx = \frac{1}{2b}\cdot\int\frac{2\cdot x}{\frac{a+bx^2}{b}}dx$

$= \frac{1}{2b}\cdot\int\frac{2\cdot x}{\frac{a}{b}+x^2}dx = \boxed{\frac{1}{2b}\cdot\ln\left|\frac{a}{b}+x^2\right|+C}$

Example

$$\int \frac{x}{5+3x^2}dx = \frac{1}{6}\ln\left|\frac{5}{3}+x^2\right| + c$$

18) $I = \int\frac{x^2}{a+bx^2}dx$

$let: x = \sqrt{\frac{a}{b}}\cdot\tan(y) \Rightarrow dx = \sqrt{\frac{a}{b}}\cdot\sec^2(y)dy$

$\Rightarrow I = \int\frac{\frac{a}{b}\cdot\tan^2(y)\cdot\sqrt{\frac{a}{b}}\cdot\sec^2(y)}{a+b(\frac{a}{b}\cdot\tan^2(y))}dy = \frac{1}{b}\cdot\int\frac{\sqrt{\frac{a}{b}}\cdot\tan^2(y)\cdot\sec^2(y)}{1+\tan^2(y)}dy$

$= \frac{1}{b}\cdot\sqrt{\frac{a}{b}}\cdot\int\tan^2(y)dy = \frac{1}{b}\cdot\sqrt{\frac{a}{b}}\cdot\int[\sec^2(y)-1]dy$

$= \frac{1}{b}\cdot\sqrt{\frac{a}{b}}\cdot[\tan(y)-y]+C = \frac{1}{b}\cdot[\sqrt{\frac{a}{b}}\cdot\tan(y)-\sqrt{\frac{a}{b}}\cdot y]+C$

$\Rightarrow \boxed{I = \frac{1}{b}\cdot[x-\sqrt{\frac{a}{b}}\cdot\tan^{-1}(\sqrt{\frac{b}{a}}\cdot x)]+C}$

Example

$$\int\frac{x^2}{3+5x^2} = \frac{1}{5}(x-\sqrt{\frac{3}{5}}\tan^{-1}\sqrt{\frac{5}{3}}x) + c$$

19) $I = \int\frac{1}{x(a+bx^2)}dx = \frac{1}{2a}\cdot\int\frac{2a}{x(a+bx^2)}dx = \frac{1}{2a}\cdot\int\frac{(2a+2bx^2)-2bx^2}{x(a+bx^2)}dx$

$= \frac{1}{2a}\cdot\int[\frac{2a+2bx^2}{x(a+bx^2)}-\frac{2bx^2}{x(a+bx^2)}]dx$

$= \frac{1}{2a}\cdot\int[\frac{2}{x}-\frac{2bx}{(a+bx^2)}]dx = \frac{1}{2a}\cdot[\ln|x^2|-\ln|a+bx^2|]+C$

$\Rightarrow \boxed{I = \frac{1}{2a}\cdot\ln\left|\frac{x^2}{a+bx^2}\right|+C}$

Example

$$\int\frac{dx}{x(3+5x^2)} = \frac{1}{6}\ln\left|\frac{x^2}{3+5x^2}\right| + c$$

20) Prove that $\int \frac{dx}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$

$$I = \int \frac{dx}{x^2(a+bx^2)} = \int \frac{(a+bx^2)-bx^2}{ax^2(a+bx^2)}dx = \int \frac{dx}{ax^2} - \int \frac{b}{a(a+bx^2)}dx = -\frac{1}{ax} - \frac{b}{a}\int \frac{dx}{a+bx^2}$$

Example

$$\int \frac{dx}{x^2(9+5x^2)} = -\frac{1}{9x} - \frac{5}{9}\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{45}}\arctan x\sqrt{\frac{5}{9}} + c$$

21) Prove that $\int \frac{dx}{(a+bx^2)^2} = \frac{x}{2a\,(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$

Solution $I = \int \frac{dx}{\left(a+bx^2\right)^2} = \frac{1}{a}\int \frac{(a+bx^2)-bx^2}{(a+bx^2)^2}dx$

$$= \underbrace{\frac{1}{a}\int \frac{dx}{a+bx^2}}_{I_1} - \underbrace{\frac{1}{a}\int \frac{bx^2}{(a+bx^2)^2}dx}_{I_2}$$

1) $I2 = \int \frac{bx^2}{(a+bx^2)^2}dx$

Integration By Parts / باستخدام التكامل بالتجزيء

$$\int udv = uv - \int vdu$$

let : $x = u\,, dx = du$

$$dv = \frac{bx}{(a+bx^2)^2}dx, \mathrm{v} = \frac{-1}{2(a+bx^2)}$$

$$\rightarrow I2 = \int \frac{bx^2}{(a+bx^2)^2}dx = \frac{-x}{2(a+bx^2)} - \int \frac{dx}{2(a+bx^2)}$$

$$\int \frac{dx}{(a+bx^2)^2} == \frac{1}{a}\int \frac{dx}{a+bx^2} - \frac{1}{a}\int \frac{bx^2}{(a+bx^2)^2}dx = \frac{1}{a}\int \frac{dx}{a+bx^2} - \frac{1}{a}\left(\left(\frac{-x}{2(a+bx^2)} + \int \frac{dx}{2(a+bx^2)}\right)\right.$$

$$= \frac{x}{2a(a+bx^2)} + \frac{1}{2a}\int \frac{dx}{a+bx^2}$$

Example

$$\int \frac{dx}{(5+2x^2)^2} = \frac{x}{10\,(5+2x^2)} + \frac{1}{10}\int \frac{dx}{5+2x^2}$$

Set(3) - المجموعة الثالثة

الدوال التي تحتوي على $\sqrt{a+bx}$

22) $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + c$

23) $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15\,b^2} + c$

24) $\int x^2\sqrt{a+bx}\,dx = -\frac{2(8a^2-12\,abx+15\,b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + c$

25) $\int \frac{x}{\sqrt{a+bx}}dx = -\frac{2(2a-bx)}{3\,b^2}\sqrt{a+bx} + c$

26) $\int \frac{x^2}{\sqrt{a+bx}}dx = \frac{2(8a^2-4abx+3\,b^2x^2)}{15\,b^2}\sqrt{a+bx} + c$

27) $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\ln\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} + c \qquad (a>0)$

28) $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}}\arctan\sqrt{\frac{a+bx}{-a}} + c \ (a<0)$

29) $\int \frac{dx}{x^2\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{ax} - \frac{b}{2a}\int\frac{dx}{x\sqrt{a+bx}}$

أنظر بعد ذلك رقم 27 أو رقم 28 Check numbers

30) $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int\frac{dx}{x\sqrt{a+bx}}$

أنظر بعد ذلك رقم 27 أو رقم 28 Check numbers

Set (3)

Solved by

Solvers	Integral
mourad24000	22
mourad24000	23
أبولونيوس	24
سيد كامل	25
laila245	26
خالد أبو محمود mourad24000	27
خالد أبو محمود mourad24000	28
hamza_mn	29
hamza_mn	30

22)Prove that $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + c$

Solution: $I = \int \sqrt{a+bx}\,dx = \frac{1}{b}\int b(a+bx)^{\frac{1}{2}}\,dx = \frac{1}{b}\frac{1}{\left(\frac{1}{2}+1\right)}(a+bx)^{\frac{1}{2}+1} + C$

$= \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C = \frac{2}{3b}\sqrt{(a+bx)^3} + C$

Example

$$\int \sqrt{3+7x}\,dx = \frac{2}{21}\sqrt{(3+7x)^3} + c$$

23)Prove that $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15\,b^2} + c$

Solution : $\boldsymbol{I} = \int x\sqrt{a+bx}\,dx = \int x(a+bx)^{\frac{1}{2}}\,dx$

let: $u = x \Longrightarrow du = dx$

$dv = \sqrt{a+bx} \Longrightarrow v = \frac{2}{3b}\sqrt{(a+bx)^3}$

$\int x\sqrt{a+bx}\,dx = \frac{2x}{3b}\sqrt{(a+bx)^3} - \frac{2}{3b}\int \sqrt{(a+bx)^3}dx$

$= \frac{2x}{3b}\sqrt{(a+bx)^3} - \frac{2}{3b.b}\int b(a+bx)^{\frac{3}{2}}\,dx$

$= \frac{2x}{3b}\sqrt{(a+bx)^3} - \frac{4}{15b^2}(a+bx)^{\frac{5}{2}} + C$

$= \frac{2x}{3b}\sqrt{(a+bx)^3} - \frac{4}{15b^2}\sqrt{(a+bx)^3}(a+bx) + C$

$= \frac{\sqrt{(a+bx)^3}}{15b^2}\big(10bx - 4(a+bx)\big) + C = \frac{\sqrt{(a+bx)^3}}{15b^2}(6bx-4a) + C = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15\,b^2} + c$

Example

$$\int x\sqrt{2+3x}\,dx = -\frac{2(4-9x)\sqrt{(2+3x)^3}}{135} + c$$

24) Prove that $\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3}+c$

Solution: Let $y^2 = a+bx$ *then* $x = \frac{y^2-a}{b}$, $x^2 = \frac{y^4-2ay^2+a^2}{b^2}$, $dx = \frac{2y}{b}dy$

Then L.H.S = $\frac{2}{b^3}\int(y^4 - 2ay^2 + a^2)(y^2)\,dy = \frac{2}{b^3}\left[\frac{y^7}{7} - \frac{2ay^5}{5} + \frac{a^2y^3}{3}\right] + c$

By substitution

$$= \frac{2y^3}{105b^3}\left[35a^2 - 42ay^2 + 15y^4\right] + c$$

$$= \frac{2\sqrt{(a+bx)^3}}{105b^3}\left[35a^2 - 42a(a+bx)^2 + 15(a+bx)^4\right] + c$$

$$= \frac{2\sqrt{(a+bx)^3}}{105b^3}\left[8a^2 - 12abx + 15b^2x^2\right] + c$$

$$= \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + c = R.H.S$$

Example

$$\int x^2\sqrt{7+11x}\,dx = -\frac{2(392-924+1825x^2)\sqrt{(7+11x)^3}}{139755} + c$$

25)Prove that $\int \frac{x}{\sqrt{a+bx}}dx = -\frac{2(2a-bx)}{3b^2}\sqrt{a+bx} + c$

Solution: $I = \int \frac{x}{\sqrt{a+bx}}dx$

but : $y^2 = a+bx$ و $x = \frac{y^2-a}{b}$, $dx = \frac{2y}{b}dy$

$$\int \frac{x}{\sqrt{a+bx}}dx = \int \frac{2}{b^2}(y^2-a)dy = \frac{2}{b^2}\left(\frac{y^3}{3} - ay\right) + c = \frac{2}{b^2}y\left(\frac{y^2}{3} - a\right) + c$$

$$= \int \frac{2}{b^2}\sqrt{a+bx}\left(\frac{a+bx-3a}{3}\right) + c$$

$$= \int \frac{2}{3b^2}\sqrt{a+bx}\,(bx-2a) + c$$

$$= \int \frac{-2}{3b^2}(2a-bx)\sqrt{a+bx} + c$$

Example

$$\int \frac{x}{\sqrt{2+3x}}dx = -\frac{2(4-3x)}{27}\sqrt{2+3x} + c$$

26)Prove that $\int \frac{x^2}{\sqrt{a+bx}}dx = \frac{2(8a^2-4abx+3\,b^2x^2)}{15\,b^2}\sqrt{a+bx}+c$

Solution: $I = \int \frac{x^2}{\sqrt{a+bx}}dx$

let : $y^2 = a+bx\,, x = \frac{1}{b}\,(y^2 - \,a\,), dx = \frac{2y}{b}dy$

$$x^2 = \frac{1}{b^2}(\,y^4 - \,2\,a\,y^2 + \,a^2)dy$$

$$I = \frac{2}{b^3}\int (\,y^4 - \,2a\,y^2 + \,a^2)dy$$

$$= \frac{2}{b^3}\left(\frac{y^5}{5} - \frac{2a\,y^3}{3} + \,a^2y\right) + \,c$$

$$= \frac{2y}{15\,b^3}(\,3y^4 - \,10\,a\,y^2 + \,15\,a^2) + \,c$$

$$= \frac{2\sqrt{a+bx}}{15\,b^3}(3(a+bx)^2 - \,10\,(\,a+bx) + 15\,a^2) + \,c$$

$$= \frac{2(8a^2-4abx+3b^2x^2)\sqrt{a+bx}}{15\,b^3} + \,c$$

Example

$$\int \frac{x^2}{\sqrt{3+5x}}dx = \frac{2(72-60x+75x^2)}{375}\sqrt{3+5x}+\,c$$

27)Prove that $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\ln\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}} + \,c$ ($a > 0$ تكون عندما)

Solution: $I \int \frac{dx}{x\sqrt{a+bx}}$

Let $u = \sqrt{a+bx} \Longrightarrow du = \frac{b}{2\sqrt{a+bx}}dx \;\Longrightarrow dx = \frac{2u}{b}du$

$$u^2 = a+bx \;\Longrightarrow x = \frac{u^2-a}{b} \Longrightarrow K = \int \frac{2du}{u^2-a}$$

If $a > 0$, we have,

$$\frac{1}{u^2-a} = \frac{1}{(u-\sqrt{a})(u+\sqrt{a})} = \frac{\frac{1}{2\sqrt{a}}}{(u-\sqrt{a})} - \frac{\frac{1}{2\sqrt{a}}}{(u+\sqrt{a})}$$

$$\Longrightarrow K = \frac{1}{\sqrt{a}}\left[\int \frac{du}{(u-\sqrt{a})} - \int \frac{du}{(u+\sqrt{a})}\right] = \frac{1}{\sqrt{a}}[ln|(u-\sqrt{a})| - ln|(u+\sqrt{a})|] + C$$

$$\Longrightarrow K = \frac{1}{\sqrt{a}}ln\left|\frac{u-\sqrt{a}}{u+\sqrt{a}}\right| + C = \frac{1}{\sqrt{a}}ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + C$$

Example

$$\int \frac{dx}{x\sqrt{6+5x}} = \frac{1}{\sqrt{6}}ln\left|\frac{\sqrt{6+5x}-\sqrt{6}}{\sqrt{6+5x}+\,6}\right| \;+\,c$$

28)Prove that Solution: I $\int \frac{dx}{x\sqrt{a+bx}} = \frac{2}{\sqrt{-a}} \arctan \sqrt{\frac{a+bx}{-a}} + c$ ($a < 0$ تكون عندما)

Solution: I $= \int \frac{dx}{x\sqrt{a+bx}}$

Let $u = \sqrt{a+bx} \Longrightarrow du = \frac{b}{2\sqrt{a+bx}} dx \Longrightarrow dx = \frac{2u}{b} du$

$u^2 = a + bx \Longrightarrow x = \frac{u^2 - a}{b}$

$\Longrightarrow K = \int \frac{2du}{u^2 - a}$

$= K = \int \frac{2du}{u^2 - a} = \int \frac{2du}{u^2 + (\sqrt{\beta})^2} = \frac{2}{\beta} \int \frac{du}{\left(\frac{u}{\sqrt{\beta}}\right)^2 + 1}$

If $a < 0$, let $\beta = -a$

Let $t = \frac{u}{\sqrt{\beta}} \Longrightarrow dt = \frac{du}{\sqrt{\beta}} \Longrightarrow du = \sqrt{\beta} dt$

$\Longrightarrow 28) = \frac{2\sqrt{\beta}}{\beta} \int \frac{dt}{t^2 + 1} = \frac{2}{\sqrt{\beta}} Arctan(t) + C$

$\Longrightarrow 28) = \frac{2}{\sqrt{-a}} Arctan\left(\frac{\sqrt{a+bx}}{\sqrt{-a}}\right) + C$

Example

$$\int \frac{dx}{x\sqrt{-8+3x}} = \frac{2}{\sqrt{8}} \arctan \sqrt{\frac{-8+3x}{8}} + c$$

29) Prove that $\int \frac{dx}{x^2\sqrt{a+bx}} = \frac{-\sqrt{a+bx}}{ax} - \frac{b}{2a} \int \frac{dx}{x\sqrt{a+bx}}$,

Solution: I $= \int \frac{dx}{x^2\sqrt{a+bx}}$

let : $y = \sqrt{a+bx}, dx = \frac{2y}{b} dy, x^2 = \left(\frac{y^2 - a}{b}\right)^2$

$I = \int \frac{2b^2 y}{(y^2 - a)^2 by} dy = \int \frac{2b}{(y^2 - a)^2} dy = 2b \int \left(\frac{c}{y - \sqrt{a}} + \frac{d}{y + \sqrt{a}}\right)^2 dy$

$c = \frac{1}{2\sqrt{a}}, d = \frac{-1}{2\sqrt{a}}$

$I = 2b \int \left(\frac{\frac{1}{2\sqrt{a}}}{y - \sqrt{a}} - \frac{\frac{1}{2\sqrt{a}}}{y + \sqrt{a}}\right)^2 dy = \frac{2b}{4a} \int \left(\frac{1}{y - \sqrt{a}} - \frac{1}{y + \sqrt{a}}\right)^2 dy$

$$=\frac{1b}{2a}\int\left(y-\sqrt{a}\right)^{-2}-\frac{2}{y^2-a}+\left(y+\sqrt{a}\right)^{-2}dy$$

$$=\frac{b}{2a}\left(\frac{-1}{y-\sqrt{a}}-\frac{2}{2\sqrt{a}}\int\left(\frac{1}{y-\sqrt{a}}-\frac{1}{y+\sqrt{a}}\right)dy+\frac{-1}{y+\sqrt{a}}\right)+c$$

$$=\frac{b}{2a}\left(\frac{-1}{y-\sqrt{a}}-\frac{1}{\sqrt{a}}\left(\ln\left|y-\sqrt{a}\right|-\ln\left|y+\sqrt{a}\right|\right)-\frac{1}{y+\sqrt{a}}\right)+c$$

Example

$$\int\frac{dx}{x^2\sqrt{6+2x}}=\frac{-\sqrt{6+2x}}{6x}+\frac{2}{72}ln\frac{6+2x}{x}+c$$

30) Prove that $\int\frac{\sqrt{a+bx}}{x}dx=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}$,

Solution: $I=\int\frac{\sqrt{a+bx}}{x}dx=\int\frac{\sqrt{a+bx}}{x}\cdot\frac{\sqrt{a+bx}}{\sqrt{a+bx}}dx$

$$=\int\frac{(a+bx)}{x\sqrt{a+bx}}dx=\int\frac{a}{x\sqrt{a+bx}}dx+\int\frac{bx}{x\sqrt{a+bx}}dx$$

$$\int b(a+bx)^{\frac{-1}{2}}+a\int\frac{dx}{x\sqrt{a+bx}}=\frac{b(a+bx)^{\frac{1}{2}}}{\frac{b}{2}}+a\int\frac{dx}{x\sqrt{a+bx}}=2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}$$

Example

$$\int\frac{\sqrt{3+2x}}{x}dx=2\sqrt{3+2x}-ln\frac{3+2x}{x}+c$$

Set(4) - المجموعة الرابعة

الدوال التي تحتوى على $\sqrt{x^2+a^2}$-

31) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} ln\left(x+\sqrt{x^2+a^2}\right) + c$

32) $\sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^2}{8} ln(x+\sqrt{x^2+a^2}) + c$

33) $x\sqrt{x^2+a^2}\,dx = \frac{\sqrt{(x^2+a^2)^3}}{3} + c$

34) $x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right) + c$

35) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) + c$

36) $\int \frac{ax}{\sqrt{(x^2+a^2)^3}}dx = \frac{x}{a^2\sqrt{x^2+a^2}} + c$

37) $\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2} + c$

38) $\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$

39) $\int \frac{x^2dx}{\sqrt{(x^2+a^2)\text{^}3}} = -\frac{x}{\sqrt{x^2+a^2}} + \ln(x+\sqrt{x^2+a^2}) + c$

40) $\int \frac{\mathrm{dx}}{\mathrm{x}\sqrt{\mathrm{x}^2+\mathrm{a}^2}} = \frac{1}{\mathrm{a}}\ln\frac{\mathrm{x}}{\mathrm{a}+\sqrt{\mathrm{x}^2+\mathrm{a}^2}} + \mathrm{c}$

41) $\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + c$

42) $\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\frac{a+\sqrt{x^2+a^2}}{x}$

43) $\int \frac{\sqrt{x^2+a^2}\,dx}{x} = \sqrt{x^2+a^2} - a\,ln\frac{a+\sqrt{x^2+a^2}}{x} + c$

44) $\int \frac{\sqrt{x^2+a^2}\,dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right) + c$

Set (4)

Solved by

رقم التكامل	المشاركون
31	سيد كامل - صادق العلي
32	mourad24000
33	خالد أبو محمود Abdullah Monla
34	hamza_mn
35	خالد أبو محمود mourad24000
36	hamza_mn - Abdullah Monla
37	عبد الواحد Abdullah Monla
38	عبد الواحد mourad24000 hamza_mn -
39	عبد الواحد mourad24000 - hamza_mn
40	عبد الواحد hamza_mn
41	عبد الواحد hamza_mn
42	عبد الواحد hamza_mn
43	عبد الواحد hamza_mn
44	عبد الواحد hamza_mn

31) Prove that $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$

$$I = \int \sqrt{x^2+a^2}\,dx = \int\left[\frac{1}{2}\cdot\sqrt{x^2+a^2} + \frac{1}{2}\cdot\sqrt{x^2+a^2}\right]dx = \int\left[\frac{1}{2}\cdot\sqrt{x^2+a^2} + \frac{x^2+a^2}{2\sqrt{x^2+a^2}}\right]dx$$

$$= \int\left[\underbrace{\frac{1}{2}\cdot\sqrt{x^2+a^2} + \frac{x^2}{2\sqrt{x^2+a^2}}}_{I1} + \underbrace{\frac{a^2}{2\sqrt{x^2+a^2}}}_{I2}\right]dx$$

$$I1 = \int\left[\frac{1}{2}\cdot\sqrt{x^2+a^2} + \frac{x^2}{2\sqrt{x^2+a^2}}\right]dx = \frac{1}{2}\cdot\int\left[1\cdot\sqrt{x^2+a^2} + x\cdot\frac{1}{2}\cdot(2x)\cdot(x^2+a^2)^{\frac{-1}{2}}\right]dx$$

now observe :

$$\left[1\cdot\sqrt{x^2+a^2} + x\cdot\frac{1}{2}\cdot(2x)\cdot(x^2+a^2)^{\frac{-1}{2}}\right] = [\frac{1}{2}\cdot x\sqrt{x^2+a^2}]'$$

$$\Rightarrow I1 = \frac{1}{2}\cdot x\sqrt{x^2+a^2} + C$$

$$I2 = \int\frac{a^2}{2\sqrt{x^2+a^2}}dx = \frac{a^2}{2}\cdot\int\frac{\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}}{x+\sqrt{x^2+a^2}}dx = \frac{a^2}{2}\cdot\int\frac{\frac{x}{\sqrt{x^2+a^2}}+1}{x+\sqrt{x^2+a^2}}dx$$

$$= \frac{a^2}{2}\cdot\int\frac{1+\frac{1}{2}\cdot(2x)\cdot(x^2+a^2)^{\frac{-1}{2}}}{x+\sqrt{x^2+a^2}}dx$$

now observe :

$$\left[\frac{1+\frac{1}{2}\cdot(2x)\cdot(x^2+a^2)^{\frac{-1}{2}}}{x+\sqrt{x^2+a^2}}\right] = [\ln|x+\sqrt{x^2+a^2}|]' \Rightarrow I2 = \frac{a^2}{2}\cdot\ln|x+\sqrt{x^2+a^2}| + C$$

$$I = I1 + I2 \Rightarrow \boxed{I = \frac{1}{2}\cdot x\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot\ln|x+\sqrt{x^2+a^2}| + C}$$

anther Solution

$$I = \int\sqrt{x^2+a^2}\,dx\ , but\ x = a\sinh\theta\ , dx = \cosh\theta\,d\theta$$

$$\therefore I = \int\sqrt{a^2\sinh^2\theta + a^2}\ .a\cosh\theta d\theta = \int a^2\cosh^2\theta\,d\theta = a^2\int\frac{1}{2}(\cosh 2\theta + 1)d\theta$$

$$= \frac{a^2}{2}.\frac{1}{2}\sinh 2\theta + \frac{a^2}{2}\theta + c = \frac{a^2}{2}.\frac{x}{a}.\frac{\sqrt{x^2+a^2}}{a} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$$

$$= \frac{x}{a}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$$

Example

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$$

32)Prove that $\int \sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^2}{8} ln\left(x+\sqrt{x^2+a^2}\right) + c$

Solution: $I = \sqrt{(x^2+a^2)^3}\,dx$

Let $x = asinh\theta \Longrightarrow dx = acosh\theta d\theta$

$\Longrightarrow I = a^4 \int cosh^4\theta d\theta$

$cosh^4\theta = \frac{cosh4\theta}{8} + \frac{cosh2\theta}{2} + \frac{3}{8}$

$$\Longrightarrow I = a^4\left[\frac{1}{8}\int cosh4\theta d\theta + \frac{1}{2}\int cosh2\theta d\theta + \frac{3}{8}\int d\theta\right] \Longrightarrow I = a^4\left[\frac{sinh4\theta}{32} + \frac{sinh2\theta}{4} + \frac{3}{8}\theta\right]$$

Now observe that

$sinh2\theta = 2sinh\theta.cosh\theta = 2\frac{x}{a}.\frac{\sqrt{a^2+x^2}}{a} = \frac{2x\sqrt{a^2+x^2}}{a^2}$

$sinh4\theta = 2sinh2\theta.cosh2\theta = 2sinh2\theta(1+2sinh^2\theta)$

$= 2\left(\frac{2x\sqrt{a^2+x^2}}{a^2}\right)\left(1+\frac{2x^2}{a^2}\right) = \frac{4x\sqrt{a^2+x^2}}{a^2} + \frac{8x^3\sqrt{a^2+x^2}}{a^4}$

$\theta = sinh^{-1}\left(\frac{x}{a}\right) = ln\left(x+\sqrt{a^2+x^2}\right)$

$\Longrightarrow I = a^4\left[\frac{x\sqrt{a^2+x^2}}{8a^2} + \frac{x^3\sqrt{a^2+x^2}}{4a^4} + \frac{x\sqrt{a^2+x^2}}{2a^2} + \frac{3}{8}ln\left(x+\sqrt{a^2+x^2}\right)\right] + C$

$I = a^4\left[\frac{5x\sqrt{a^2+x^2}}{8a^2} + \frac{x^3\sqrt{a^2+x^2}}{4a^4} + \frac{3}{8}ln\left(x+\sqrt{a^2+x^2}\right)\right] + C$

$I = \frac{x}{8}\sqrt{a^2+x^2}(5a^2+2x^2) + \frac{3}{8}a^4 ln\left(x+\sqrt{a^2+x^2}\right) + C$

Example

$$\sqrt{(x^2+a^2)^3}\,dx = \frac{x}{8}(2x^2+5a^2)\sqrt{x^2+a^2} + \frac{3a^2}{8}ln(x+\sqrt{x^2+a^2}) + c$$

33)Prove that $\int x\sqrt{x^2+a^2}\,dx = \frac{\sqrt{(x^2+a^2)^3}}{3} + c$

Solution $I = \int x\sqrt{x^2+a^2}\,dx = \frac{1}{2}\int 2x\sqrt{x^2+a^2}\,.dx = \frac{1}{2}.\frac{2}{3}\sqrt{(x^2+a^2)^3} + c = \frac{1}{3}\sqrt{(x^2+a^2)^3} + c$

anther Solution $\int x\sqrt{x^2+a^2}\,dx$

let : $x = a\tan^{-1}y$ و $dx = a\sec^2 y\,dy$, $y = \tan^{-1}\frac{x}{a}$

$$\int x\sqrt{x^2+a^2}\,dx = \int a^2\sec y\tan y\,(a\sec^2 y)dy = a^3\int \sec y\tan y\,(\sec^2 y)\,dy$$

$$= \frac{a^3\sec y}{3} + c = \frac{a^3(x^2+a^2)^{\frac{3}{2}}}{3a^3} + c = \frac{(x^2+a^2)^{\frac{3}{2}}}{3} + c$$

Example

$$\int x\sqrt{x^2+8^2}\,dx = \frac{\sqrt{(x^2+64)^3}}{3} + c$$

34) Prove that $\int x^2\sqrt{x^2+a^2}\,dx = \frac{x}{8}(2x^2+a^2)\sqrt{x^2+a^2} - \frac{a^4}{8}\ln\left(x+\sqrt{x^2+a^2}\right)+c$

Solution: $I = \int x^2\sqrt{x^2+a^2}\,dx$

$v = x\,, dv = dx\,, du = x(x^2+a^2)^{\frac{1}{2}}, u = \frac{2}{3}(x^2+a^2)^{\frac{3}{2}}$

$\therefore \int vdu = vu - \int u\,dv = \frac{2}{3}x(x^2+a^2)^{\frac{3}{2}} - \frac{2}{3}\int\sqrt{(x^2+a^2)^3}\,dv$

Example

$$\int x^2\sqrt{x^2+9}\,dx = \frac{x}{8}(2x^2+9)\sqrt{x^2+9} - \frac{81}{8}\ln\left(x+\sqrt{x^2+64}\right)+c$$

35) Prove that $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2})+c$

Solution: $I = \int \frac{dx}{\sqrt{x^2+a^2}}$

$Let\ \sqrt{x^2+a^2} = z - x \Rightarrow x^2+a^2 = z^2-2zx+x^2$

$\Rightarrow \boxed{x = \frac{z^2-a^2}{2z}},\quad \boxed{z - x = \frac{z^2+a^2}{2z}}$

$\boxed{dx = \frac{z^2+a^2}{2z^2}dz}$

$\Rightarrow I = \int\left(\frac{z^2+a^2}{2z^2}\right)\left(\frac{2z}{z^2+a^2}\right)dz = \int\frac{dz}{z}$

$\Rightarrow I = ln|z| + C = ln\left(\sqrt{x^2+a^2}+x\right)+C$

Example

$$\int \frac{dx}{\sqrt{x^2+121}} = \ln(x+\sqrt{x^2+121})+c$$

36) Prove that $\int \frac{ax}{\sqrt{(x^2+a^2)^3}}dx = \frac{x}{a^2\sqrt{x^2+a^2}} + c$

Solution: $I = \int \frac{ax}{\sqrt{(x^2+a^2)^3}} = \int \frac{dx}{(x^2+\ a^2)^{\frac{3}{2}}} = \int \frac{dx}{(x^2\left(1+\frac{a^2}{x^2}\right))^{\frac{3}{2}}} = \int \frac{dx}{x^3\left(\ 1+\frac{a^2}{x^2}\right)^{\frac{3}{2}}}$

$let: 1+\frac{a^2}{x^2} = y$ و $dx = \frac{-x^3}{2a^2}dy$

$I = \int \frac{-x^3 dy}{2a^2x^3y^{\frac{3}{2}}} = \frac{-1}{2a}\int y^{\frac{-3}{2}}dy = \frac{-y^{\frac{-1}{2}}}{-\mathrm{a}^2} + \mathrm{c} = \frac{1}{a^2\sqrt{y}} = \frac{1}{a^2\sqrt{1+\frac{a^2}{x^2}}} + c = \frac{x}{a^2\sqrt{x^2+\ a^2}} + c$

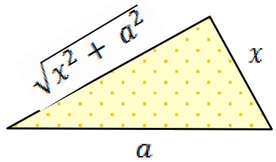
anther Solution

let: $x = a\tan y$ و $dx = a\sec^2 y\, dy$

$$\int \frac{ax}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{\mathrm{dx}}{(\mathrm{x}^2+\mathrm{a}^2)^3} = \int \frac{a\sec^2 y}{a^3\sec^3 y}dy = \frac{1}{a^2}\int \frac{1}{\sec y}dy = \frac{1}{a^2}\int \cos y\, dy = \frac{\sin y}{a^2} + c$$

$\sin y = \frac{x}{\sqrt{x^2+a^2}}$

$I = \frac{x}{a^2\sqrt{x^2+a^2}} + c$



Example

$$\int \frac{ax}{\sqrt{(x^2+25)^3}}dx = \frac{x}{25\sqrt{x^2+25}} + c$$

37) Prove that $\int \frac{x\, dx}{\sqrt{x^2+\ a^2}} = \sqrt{x^2+\ a^2} + c$

Solution : *let* : $(\,x^2+\ a^2) = y$ و $x = \sqrt{y-a^2}$ و $dx = \frac{dy}{2\sqrt{y-a^2}}$

$I = \int \frac{x\, dx}{\sqrt{x^2+\ a^2}} = \int \frac{\sqrt{y-a^2}}{2\sqrt{y}\sqrt{y-a^2}}\, dy\ = \int \frac{dy}{2\sqrt{y}} = \sqrt{y} + c = \sqrt{x^2+\ a^2} + c$

another Solution

let : $x = a\tan y$ و $dx = a\sec^2 y dy$ و $\sqrt{x^2+\ a^2} = \sqrt{a^2\tan^2 y + a^2} = \sqrt{a^2(\tan^2 y + 1\,)}$

$= a\sqrt{\sec^2 y}\ = a\ secy$

$I = \int \frac{x\, dx}{\sqrt{x^2+\ a^2}} = \int \frac{a\tan y\,.\,a\sec^2 y}{\mathrm{a}\sec \mathrm{y}}\mathrm{dy} = \int \mathrm{a}\tan \mathrm{y} \sec \mathrm{y}\, \mathrm{dy} = \mathrm{a}\sec \mathrm{y} + \mathrm{c} = \sqrt{\mathrm{x}^2+\ \mathrm{a}^2} + \ \mathrm{c}$

another Solution

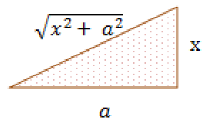
$$\int \frac{x\,dx}{\sqrt{x^2+a^2}} = \int \frac{2x\,dx}{2\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}+c$$

Example

$$\int \frac{x\,dx}{\sqrt{x^2+49}} = \sqrt{x^2+49}+c$$

38) Prove that $\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+c$

Solution : *let*: $x = a\tan y$ و $dx = a\sec^2 y\,dy$



$$I = \int \frac{x^2dx}{\sqrt{x^2+a^2}} = \int \frac{a^2\tan^2 y.\,a\sec^2 y}{\sqrt{a^2\tan^2 y+a^2}}dy = \int \frac{a^3\sec^2 y\tan^2 y}{\sqrt{a^2\sec^2 y}}dy$$

$$= \int a^2 secy\tan^2 y\,dy = a^2\int secy\,(\sec^2 y-1)\,dy = \int a^2\sec^3 y\,dy - a^2\int secy\,dy$$

$u = a^2 secy$, $du = a^2 secy\tan y\,dy$

$dv = \sec^2 y\,dy$, $\quad v = tany$

$$I = \int a^2\sec^3 dy = a^2\sec y\,tan\,y - \int a^2\sec y\tan^2 y\,dy$$

$$= a^2\sec y\,tan\,y - \int a^2\sec y(\sec^2 y-1)\,dy = a^2\sec y\tan y - a^2\int \sec^3 y + \int a^2\sec y\,dy$$

$$= \frac{1}{2}a^2 sec\,y\,tan\,y + \frac{1}{2}\int a^2\sec y\,dy$$

$$I = \frac{1}{2}a^2 sec\,y\,tan\,y + \frac{1}{2}\int a^2\,sec\,y\,dy - \int a^2\,sec\,y\,dy$$

$$= \frac{1}{2}a^2\,sec\,y\,tan\,y\,dy - \frac{1}{2}a^2\int \frac{sec\,y\,(sec\,y+tan\,y)}{sec\,y+tan\,y}dy$$

$$= \frac{1}{2}a^2 secy\,tan\,y - \frac{1}{2}a^2\,ln\,|\,secy+tan\,y\,|+c$$

$$= \frac{1}{2}a^2\frac{\sqrt{x^2+a^2}}{x} - \frac{1}{2}\;a^2 ln\left|\frac{\sqrt{x^2+a^2}}{x}+\frac{x}{a}\right|+c$$

38 – *another* Solution

$$\int \frac{x^2dx}{\sqrt{x^2+a^2}}$$

Integration By Parts - باستخدام التكامل بالتجزيئ

$$u = x\,,du = dx\,,dv = \frac{x}{\sqrt{x^2+a^2}}dx\,.v = \sqrt{x^2+a^2}$$

$$\int \frac{x^2dx}{\sqrt{x^2+a^2}} = x\sqrt{x^2+a^2} - \underbrace{\int \sqrt{x^2+a^2}\,dx}_{31} = x\sqrt{x^2+a^2} - \frac{x}{2}\sqrt{x^2+a^2} - \frac{a^2}{2}\ln(x+\sqrt{x^2+a^2}) + c$$

$$\int \frac{x^2dx}{\sqrt{x^2+a^2}} = \frac{x\sqrt{x^2+a^2}}{2} - \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + c$$

another Solution

$$\int \frac{x^2dx}{\sqrt{a^2+x^2}} = \int x\frac{xdx}{\sqrt{a^2+x^2}} = I$$

Integration By Parts -باستخدام التكامل بالتجزيء

Let $u = x \longrightarrow du = dx,$

$$dv = \frac{xdx}{\sqrt{a^2+x^2}} \longrightarrow v = \sqrt{a^2+x^2}$$

$$\Longrightarrow I = x\sqrt{a^2+x^2} - \underbrace{\int \sqrt{a^2+x^2}dx}_{Intégrale\ (31)}$$

$$\Longrightarrow I = x\sqrt{a^2+x^2} - \left(\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}ln\left(x+\sqrt{a^2+x^2}\right)\right) + C$$

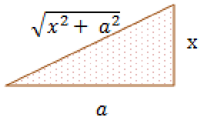
$$\Longrightarrow \boxed{\int \frac{x^2dx}{\sqrt{a^2+x^2}} = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}ln\left(x+\sqrt{a^2+x^2}\right) + C}$$

Example

$$\int \frac{x^2dx}{\sqrt{x^2+16}} = \frac{x}{2}\sqrt{x^2+16} - 8\ln\left(x+\sqrt{x^2+18}\right) + c$$

39) Prove that $\int \frac{x^2dx}{\sqrt{(x^2+\ a^2)^3}} = -\frac{x}{\sqrt{x^2+\ a^2}} + \ln(x+\sqrt{x^2+\ a^2})\ +\ c$

let : $x = a\tan y$ و $dx = a\sec^2 y\, dy$



$$I = \int \frac{a^2\tan^2 y\,.\,a\sec^2 y}{\sqrt{(a^2\tan^2 y + a^2)^3}}dy = \int \frac{a^3\sec^2 y\tan^2 y}{\sqrt{a^6\sec^6 y}}dy =$$

$$= \int \tan^2 y\cos y\, dy = \int \frac{\sin^2 y\cos y}{\cos^2 y}\, d\,y = \int \frac{\sin^2 y}{\cos y}\, d\,y$$

let : $\sin y = s\,,dy = \frac{ds}{\cos y}$

$$I = a\int \frac{s^2}{1-s^2}ds = -a\int \frac{1-1-s^2}{1-s^2}\,ds = -a\int \frac{1-s^2}{1-s^2}ds + a\int \frac{1}{1-s62}.\frac{2}{2}\,ds$$

$$= -as\ + \frac{1}{2}a\int \frac{1}{1-s} - \frac{1}{1+s}ds = -a\,s + \frac{1}{2}a\,(\ln|\,1-s| - \ln|1+s|ds + c$$

$$= \frac{-a\,x}{\sqrt{x^2+\ a^2}}.\frac{1}{2}(\ln\left|\,1-\frac{x}{\sqrt{x^2+\ a^2}}\right| - \ln\left|\,1+\frac{x}{\sqrt{x^2+\ a^2}}\right| +\ c$$

another Solution

$$\int \frac{x^2dx}{\sqrt{(x^2+\ a^2)}^3} = \int \frac{x^2+a^2-a^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{dx}{\sqrt{x^2+\ a^2}} - a^2\int \frac{dx}{\sqrt{(x^2+\ a^2)^3}}$$

$$= ln\left(\,x+\sqrt{x^2+\ a^2}\right) - \ a^2.\frac{x}{a^2\sqrt{x^2+\ a^2}} +\ c$$

$$= \frac{-x}{\sqrt{x^2+\ a^2}} + \ln\left(\,x+\sqrt{x^2+\ a^2}\right) + c$$

another Solution

$\int \frac{x^2}{\sqrt{(a^2+x^2)^3}}dx = \int x\frac{xdx}{\sqrt{(a^2+x^2)^3}} = I$ *Let*: $u = x \longrightarrow du = dx,$

$$dv = \frac{xdx}{\sqrt{(a^2+x^2)^3}} \longrightarrow v = \frac{-1}{\sqrt{a^2+x^2}}$$

$$\Longrightarrow I = \frac{-x}{\sqrt{a^2+x^2}} + \underbrace{\int \frac{dx}{\sqrt{a^2+x^2}}}_{Intégrale\ (35)}$$

$$\Longrightarrow \boxed{\int \frac{x^2}{\sqrt{(a^2+x^2)^3}} = \frac{-x}{\sqrt{a^2+x^2}} + ln\left(x+\sqrt{a^2+x^2}\right) + C}$$

Example

$$\int \frac{x^2dx}{\sqrt{(x^2+\ 4)^3}} = -\frac{x}{\sqrt{x^2+\ 4}} + \ln(x+\sqrt{x^2+\ 4}\,)\ +\ c$$

40) Prove that $\int \frac{dx}{x\sqrt{x^2 + a^2}} = \frac{1}{a} ln \frac{x}{a + \sqrt{x^2 + a^2}} + c$

$let : y = \sqrt{x^2 + a^2} , dx = \frac{y}{x} dy$

$i = \int \frac{1}{xy} . \frac{y}{x} dy = \int \frac{1}{x^2} dy = \int \frac{dy}{y^2 - a^2}$

$\frac{1}{y^2 - a^2} = \frac{u}{y - a} + \frac{v}{y + a}$ و $u = \frac{1}{2a} , v = \frac{-1}{2a}$

$I = \int \left(\frac{\frac{1}{2a}}{y - a} - \frac{\frac{1}{2a}}{y + a} \right) dy = \frac{1}{2a} (\ln|y - x| - \ln|y + a|) + c$

another Solution

$I = \frac{dx}{x\sqrt{x^2 + a^2}}$

$let : u = \frac{1}{x} , du = \frac{1}{x^2} dx , dx = -x^2 du , x^2 + a^2 = \frac{1}{u^2} + a^2 = \frac{1 + a^2u^2}{u^2}$

$\frac{-dx}{x\sqrt{x^2 + a^2}} = \int \frac{-x^2 du}{x\sqrt{\frac{1 + a^2u^2}{u^2}}} = -\int \frac{x\, du}{\sqrt{\frac{1 + a^2u^2}{u^2}}} du = \int \frac{\frac{-1}{u}}{\sqrt{\frac{1 + a^2u^2}{u^2}}} = -\int \frac{du}{\sqrt{1 + a^2u^2}}$

let $: t = a\,u , dt = a\, du$

$I = -\int \frac{du}{\sqrt{1 + (au)^2}} = -\frac{1}{a} \int \frac{dt}{\sqrt{1 + t^2}} = -\frac{1}{a} \ln(t + \sqrt{t^2 + 1}) + c$

$t = au = \frac{a}{x}$

$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a} ln(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 + 1} + c = -\frac{1}{a} \ln(\frac{1}{x}\left(a + \sqrt{a^2 + x^2}\right) + c = \frac{1}{a} \ln \frac{x}{a + \sqrt{a^2 + x^2}} + c$

Example

$$\int \frac{dx}{x\sqrt{x^2 + 9}} = \frac{1}{3} ln \frac{x}{3 + \sqrt{x^2 + 9}} + c$$

41) Prove that $\int \frac{dx}{x^2\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{a^2x} + c$

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = \int \frac{dx}{x^2\sqrt{x^2\left(1+\frac{a^2}{x^2}\right)}} = \int \frac{dx}{x^3\sqrt{\left(1+\frac{a^2}{x^2}\right)}}$$

$$let: y = \sqrt{1+\frac{a^2}{x^2}}, y^2 = 1+\frac{a^2}{x^2}, dx = \frac{-x^3y\,dy}{a^2}$$

$$i = \int \frac{-x^3y\,dy}{a^2x^3y} = \int \frac{-1}{a^2}dy = \frac{-y}{a^2} + c = \frac{-\sqrt{x^2+a^2}}{a^2x} + c$$

another Solution

$$I = \int \frac{dx}{x^2\sqrt{x^2+a^2}}$$

$$let: u = \frac{a}{x}, du = -\frac{a}{x^2}dx$$

$$u^2 = \frac{a^2}{x^2}, x^2 + a^2 = \frac{a^2}{u^2} + a^2 = \frac{a^2+a^2u^2}{u^2}$$

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = \int \frac{-du}{\sqrt{\frac{a^2+a^2u^2}{u^2}}} = -\frac{1}{a^2}\int \frac{u\,du}{\sqrt{1+u^2}} = -\frac{1}{a^2}\sqrt{1+u^2} + c = -\frac{1}{a^2}\sqrt{1+\frac{a^2}{x^2}} + c$$

$$= -\frac{\sqrt{x^2+a^2}}{a^2x} + c$$

Example

$$\int \frac{dx}{x^2\sqrt{x^2+16}} = -\frac{\sqrt{x^2+16}}{16x} + c$$

42) Prove that $\int \frac{dx}{x^3\sqrt{x^2+a^2}} = -\frac{\sqrt{x^2+a^2}}{2a^2x^2} + \frac{1}{2a^3}\ln\frac{a+\sqrt{x^2+a^2}}{x}$

$let : y = \sqrt{x^2+a^2}$ و $y^2 = x^2+a^2$ و $dx = \frac{y}{x}dy$

$$I = \int \frac{1}{x^3 y}.\frac{y}{x}dy = \int \frac{1}{x^4}dy = \int \frac{dy}{x^4} = \int \frac{dy}{y^2-a^2} = \int \left(\frac{1}{y^2-a^2}\right)^2 dy$$

بالتجزيء- Integration By Parts

$$\frac{1}{(y^2-a^2)^2} = \frac{u}{y-a} + \frac{v}{y+a}, u(y+a) + v(y-a) = 1, u = \frac{1}{a}, v = \frac{-1}{2a}$$

$$I = \int \left(\frac{\frac{1}{2a}}{y-a} - \frac{\frac{1}{2a}}{y+a}\right)dy = \frac{1}{4a^2}\int (y-a)^{-2} - \frac{2}{(y-a)(y+a)} + (y+a)^{-2}$$

$$= \frac{1}{4a^2}\left(\frac{-2}{y-a} - \frac{1}{a}\int \frac{1}{y-a} - \frac{1}{y+a} - \frac{1}{y+a}\right) + c$$

$$= \frac{1}{4a^2}\left(\frac{-2}{y-a} - \frac{1}{a}\ln\left|\frac{y-a}{y+a}\right| - \frac{1}{y+a}\right) + c$$

another Solution

$$\text{I} = \int \frac{dx}{x^3\sqrt{x^2+a^2}}$$

$$u = \frac{1}{x}, du = \frac{-dx}{x^2}, dv = \frac{dx}{x^3\sqrt{x^2+a^2}}, v = \frac{-\sqrt{x^2+a^2}}{a^2x}$$

$$I = \int \frac{dx}{x^3\sqrt{x^2+a^2}} = \frac{-\sqrt{x^2+a^2}}{a^2x^2} - \int \frac{\sqrt{x^2+a^2}}{a^2x}du = \frac{-\sqrt{x^2+a^2}}{a^2x^2} - \frac{1}{a^2}\left(\int \frac{dx}{x\sqrt{x^2+a^2}} + \int \frac{a^2dx}{x^3\sqrt{x^2+a^2}}\right)$$

$$= \underbrace{\frac{-\sqrt{x^2+a^2}}{a^2x^2} - \frac{1}{a^2}\int \frac{dx}{x\sqrt{x^2+a^2}}}_{40} - \int \frac{dx}{x^3\sqrt{x^2+a^2}}$$

$$I = \frac{-\sqrt{x^2+a^2}}{2a^2x^2} - \frac{1}{2a^2}\left(\frac{1}{a}ln\frac{x}{a+\sqrt{x^2+a^2}}\right) + c = \frac{-\sqrt{x^2+a^2}}{2a^2x^2} - \frac{1}{2a^3}\left(ln\frac{x}{a+\sqrt{x^2+a^2}}\right) + c$$

Example

$$\int \frac{dx}{x^3\sqrt{x^2+1}} = -\frac{\sqrt{x^2+1}}{2x^2} + \frac{1}{2}\ln\frac{1+\sqrt{x^2+1}}{x}$$

43) Prove that $\int \frac{\sqrt{x^2+a^2}\,dx}{x} = \sqrt{x^2+a^2} - \mathrm{a}\ln\frac{\mathrm{a}+\sqrt{\mathrm{x}^2+\mathrm{a}^2}}{\mathrm{x}} + \mathrm{c}$

$let : y^2 = a^2 + x^2, dx = \frac{y}{x}dy$

$$I = \int \frac{y^2}{x^2}dy = \int \frac{y^2}{y^2-a^2}dy = \int \frac{y^2-a^2+a^2}{y^2-a^2}dy = \int \frac{y^2-a^2}{y^2-a^2}dy + \int \frac{a^2}{y^2-a^2}\cdot\frac{2}{2}\,d\,y$$

$$= y + \frac{1}{2}\int\left(\frac{a}{y-a} - \frac{a}{y+a}\right)dy = y + \frac{1}{2a}(\ln|y-a| - \ln|\,y+a|) + c$$

$$= \sqrt{x^2+a^2} + \frac{1}{2\mathrm{a}}\ln\left|\frac{\sqrt{\mathrm{x}^2+\mathrm{a}^2}-\mathrm{a}}{\sqrt{\mathrm{x}^2+\mathrm{a}^2}+\mathrm{a}}\right| + \mathrm{c}$$

another Solution

$$I = \int \frac{x^2+a^2}{x\sqrt{x^2+a^2}}\,dx = \underbrace{\int \frac{x\,dx}{\sqrt{x^2+a^2}}}_{37} + a^2\underbrace{\int \frac{dx}{x\sqrt{x^2+a^2}}}_{40} = \sqrt{x^2+a^2} + a^2\left(\frac{1}{a}ln\frac{x}{a+\sqrt{x^2+a^2}}\right) + c$$

$$= \sqrt{x^2+a^2} + a\left(ln\frac{x}{a+\sqrt{x^2+a^2}}\right) + c$$

Example

$$\int \frac{\sqrt{x^2+1}\,dx}{x} = \sqrt{x^2+1} - \ln\frac{1+\sqrt{\mathrm{x}^2+1}}{\mathrm{x}} + \mathrm{c}$$

44) Prove that $\int \frac{\sqrt{x^2+a^2}\,dx}{x^2} = -\frac{\sqrt{x^2+a^2}}{x} + \ln\left(x+\sqrt{x^2+a^2}\right) + c$

let : $x = a\tan y$ و $dx = a^2\sec^2 y\,dy$

$$I = \frac{\sqrt{a^2\tan^2 y + a^2}}{a^2\tan^2 y}\cdot a^2\sec^2 y\,dy = \int \frac{a\sec^2 y}{\tan^2 y}dy = a\int \frac{\cos^2 y}{\sin^2 y}\cdot\frac{1}{\cos^3 y}dy$$

$$= \int a\cdot\frac{1}{\sin^2 y}\cdot\frac{1}{cosy}dy = a\int \sec y\,\csc^2 y\,dy =$$

بالتجزيئ /

$u = a\sec y\;, du = a\sec y\tan y\,dy\,, dv = \csc^2 y\,dy\,, v = -\cot y$

$$I = -a\sec y\cot y + \int a\sec y\tan y\cot y\,dy = -a\csc y + a\int \sec y\,\frac{(\sec y+\tan y)}{(\sec y+\tan y)}\,dy$$

$$= -a\sec y + a\ln|\sec y + \tan y| + c = \frac{-a\sqrt{x^2+a^2}}{x} + a\,ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + c$$

44 – *another* Solution

$$I = \int \frac{x^2 + a^2}{x^2\sqrt{x^2 + a^2}}dx = \int \frac{dx}{\sqrt{x^2 + a^2}} + a^2 \int \frac{dx}{x^2\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + a^2\left(\frac{-\sqrt{x^2 + a^2}}{a^2 x}\right) + c$$

$$= \ln\left(x + \sqrt{x^2 + a^2}\right) + \left(\frac{-\sqrt{x^2 + a^2}}{x}\right) + c$$

Example

$$\int \frac{\sqrt{x^2 + 64}\,dx}{x^2} = -\frac{\sqrt{x^2 + 64}}{x} + \ln\left(x + \sqrt{x^2 + 64}\right) + c$$

Set(5) - المجموعة الخامسة

الدوال التي تحتوي على $\sqrt{a^2 - x^2}$

45) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

46) $\int \frac{dx}{\sqrt{a^2 - x^2}} = arc\sin\frac{x}{a} + c$

47) $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + c$

48) $\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + c$

49) $\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}} + c$

50) $\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\,arc\sin\frac{x}{a} + c$

51) $\int \sqrt{a^2 - x^2}dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}arc\sin\frac{x}{a} + c$

52) $\int \sqrt{(a^2 - x^2)^3}dx = \frac{x}{8}(5\,a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8}arc\sin\frac{x}{a} + c$

53) $\int x\sqrt{a^2 - x^2}\,dx = -\frac{\sqrt{a^2 - x^2}}{3} + c$

54) $\int x\sqrt{(a^2 - x^2)^3}\,dx = -\frac{\sqrt{(a^2 - x^2)^3}}{3} + c$

55) $\int x^2\sqrt{a^2 - x^2}\,dx = \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^2}{8}arc\sin\frac{x}{a} + c$

56) $\int \frac{x^2dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{\sqrt{a^2 - x^2}} - arc\sin\frac{x}{a} + c$

57) $\int \frac{dx}{x\sqrt{a^2 - x^2}} = \frac{1}{a}ln\frac{x}{a + \sqrt{a^2 - x^2}} + c$

Set (5)

Solved by

المشاركون	رقم التكامل
عبد الواحد hamza_mn	45
عبد الواحد hamza_mn	46
hamza_mn - mourad24000	47
خالد أبو محمود - عبد الواحد hamza_mn	48
خالد أبو محمود - عبد الواحد hamza_mn	49
عبد الواحد hamza_mn	50
عبد الواحد hamza_mn	51
عبد الواحد hamza_mn	52
خالد أبو محمود - عبد الواحد hamza_mn	53
خالد أبو محمود - عبد الواحد hamza_mn	54
عبد الواحد hamza_mn	55
عبد الواحد hamza_mn	56
خالد أبو محمود - عبد الواحد hamza_mn	57
خالد أبو محمود - عبد العاطي وفا hamza_mn	58
عبد العاطي وفا hamza_mn	59
عبد العاطي وفا hamza_mn	60
عبد العاطي وفا hamza_mn	61

45) Prove that $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$

let : $x = \sin y \;\; , dx = \cos y \, dy$

$$I = \int \frac{\cos y \, dy}{\sqrt{1-\sin^2 y}} = \int \frac{\cos y \, dy}{\sqrt{\cos^2 y}} = \int \frac{\cos y}{\cos y} dy = y + c = \arcsin x + c$$

Example

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + c$$

46) Prove that $\int \frac{dx}{\sqrt{a^2 - x^2}} = arc \sin \frac{x}{a} + c$

let: $x = a \sin y$ و $a \cos y \, dy$

$$I = \frac{a \cos y \, dy}{\sqrt{a^2 - a^2 \sin^2 y}} = \int \frac{a \cos y}{\sqrt{a^2 \cos^2 y}} dy = \int \frac{a \cos y}{a \sin y} dy = y + c = arc \sin \frac{x}{a} + c$$

Example

$$\int \frac{dx}{\sqrt{100 - x^2}} = arc \sin \frac{x}{10} + c$$

47) Prove that $\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \frac{x}{a^2\sqrt{a^2 - x^2}} + c$

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = \int \frac{dx}{(a^2 - x^2)^{\frac{3}{2}}} = \int \frac{dx}{x^2 \left(\frac{a^2}{x^2} - 1\right)^{\frac{3}{2}}} = \int \frac{dx}{x^3 \left(\frac{a^2}{x^2} - 1\right)^{\frac{3}{2}}}$$

let : $\frac{a^2}{x^2} - 1 = y$ و $dx = \frac{-x^3}{2a^2} dy$

$$I = \int \frac{1}{x^3 \, y^{\frac{3}{2}}} \cdot \frac{-x^3}{2a^2} dy = \frac{-1}{2a^2} \int y^{\frac{-3}{2}} dy = \frac{-1}{2a} \cdot \frac{1}{\sqrt{\frac{a^2}{x^2} - 1}} + c = \frac{1}{a^2} \cdot \frac{1}{\sqrt{\frac{a^2}{x^2} - 1}} + c = \frac{x}{a^2\sqrt{a^2 - x^2}} + c$$

anther Solution

$$\int \frac{dx}{\sqrt{(a^2 - x^2)^3}} = I$$

Let $x = a.\sin t \Longrightarrow t = Arcsin\left(\frac{x}{a}\right) \quad and \; dx = a.cost$

$x = a.\sin t$, $\sqrt{a^2 - x^2} = a.\cos t \Rightarrow \boxed{\tan t = \dfrac{x}{\sqrt{a^2 - x^2}}}$

Now,

$$I = \int a\cos t\,(a^2 - a^2\sin^2 t)^{\frac{-3}{2}} dt = \frac{1}{a^2}\int \frac{dt}{\cos^2 t}$$

$$\Rightarrow I = \frac{1}{a^2}\int d(\tan t) = \frac{1}{a^2}\tan t + C = \frac{1}{a^2}\frac{x}{\sqrt{a^2 - x^2}} + C$$

Example

$$\int \frac{dx}{\sqrt{(9 - x^2)^3}} = \frac{x}{9\sqrt{9 - x^2}} + c$$

48) Prove that $\displaystyle\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} + c$

let : $u = \sqrt{a^2 - x^2}$, $du = \dfrac{-2x\,dx}{2\sqrt{a^2 - x^2}}dx$, $dx = \dfrac{2\,u\,du}{-2\,x}$

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = \int \frac{x}{4}.\frac{2u\,du}{-2x} = \int -du = -u + c = \sqrt{a^2 - x^2} + c$$

Example

$$\int \frac{x\,dx}{\sqrt{16 - x^2}} = -\sqrt{16 - x^2} + c$$

49) Prove that $\displaystyle\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \frac{1}{\sqrt{a^2 - x^2}} + c$

let $u = a^2 - x^2$, $du = -2x\,dx$, $dx = \dfrac{du}{-2x}$

$$\int \frac{x\,dx}{\sqrt{(a^2 - x^2)^3}} = \int \frac{x\,du}{u^{\frac{3}{2}}.-2x} = -\frac{1}{2}\int u^{\frac{-3}{2}}du = -\frac{1}{2}\frac{u^{\frac{-1}{2}}}{\frac{-1}{2}} + c = \frac{1}{\sqrt{a^2 - x^2}} + c$$

anther Solution

$$\int \frac{x\,dx}{\sqrt{a^2 - x^2}}$$

let : $(a^2 - x^2)^{\frac{3}{2}} = y$ و $a^2 - x^2 = y^{\frac{2}{3}}$ و $-2x\,dx = \dfrac{2}{3}y^{\frac{-1}{3}}dy$ و $dx = \dfrac{-dy}{3\,y^{\frac{1}{3}}x}$

$$I = \int \frac{x}{y}.\frac{-dy}{3\,y^{\frac{1}{3}}x} = \frac{-1}{3}\int y^{\frac{-2}{3}}\,dy = \frac{-1}{3}.\frac{y^{\frac{-1}{3}}}{\frac{-1}{3}} = \frac{1}{\sqrt[3]{\sqrt{(a^2 - x^2)^3}}} + c = \frac{1}{\sqrt{a^2 - x^2}} + c$$

Example

$$\int \frac{x\,dx}{\sqrt{(16-x^2)^3}} = \frac{1}{\sqrt{16-x^2}} + c$$

50) Prove that $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\, arc\sin\frac{x}{a} + c$

$u = x$ و $du = dx$, $dv = \frac{x}{\sqrt{a^2-x^2}}dx$ و $v = -\sqrt{a^2-x^2}$

$$I = -x\sqrt{a^2-x^2} + \underbrace{\int \sqrt{a^2-x^2}\,dx}_{51}$$

$$= -x\sqrt{a^2-x^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c = \frac{a^2}{2}\cos^{-1}\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + c$$

anther Solution

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{x^2 dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} =$$

$u = arc\sin\frac{x}{a}$ و $\frac{x}{a}\sin u$ و $du = a\cos u\,du$, $1-\sin^2 u = \cos^2 u$

$$I = \frac{1}{a}\int \frac{a^2\sin^2 u\,.\,a\cos u\,du}{\sqrt{\cos^2 u}} = a^2\int \sin^2 u\,du = a^2\int \frac{1-\cos 2u}{2}\,du = \frac{a^2 u}{2} - \frac{a^2}{2}\int \cos 2u\,du$$

$$= \frac{a^2 u}{2} - \frac{a^2}{2}.\frac{\sin 2u}{2} + c = \frac{a^2 arc\sin\frac{x}{a}}{2} - \frac{a^2}{2}\sqrt{1-\left(\frac{u}{a}\right)^2}\,.\frac{x}{a} + c = \frac{a^2 arc\sin\frac{x}{a}}{2} - \frac{ax}{2}\sqrt{\frac{a^2-x^2}{a^2}} + c$$

$$= -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\,arc\sin\frac{x}{a} + c$$

Example

$$\int \frac{x^2\,dx}{\sqrt{25-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\,arc\sin\frac{x}{a} + c$$

51)Prove that $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}arc\sin\frac{x}{a} + c$

let : $x = a\sin y$ و $dx = a\cos y\,dy$

$$I = \int \sqrt{a^2 - a\%2\sin y}\;.a\cos 2y\;dy = \int \sqrt{a^2(1-\sin^2 y)}\;.a\cos y\,dy = \int a^2\cos^2 y\,dy$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \sin y\cos y\right) + c = \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) + c = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c =$$

anther Solution

By Parts - بالتجزيئ

$$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}}$$

$u = x$ و $du = dx\,, dv = \dfrac{x}{\sqrt{a^2 - x^2}}$ و $v = -\sqrt{a^2 - x^2}$

$$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -x\sqrt{a^2 - x^2} + \int \sqrt{a^2 - x^2}\, du$$

$$\int \sqrt{a^2 - x^2}\, du = \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} + x\sqrt{a^2 - x^2} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} arc\sin\frac{u}{a} + x\sqrt{a^2 - x^2} + c$$

$$I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} arc\sin\frac{u}{a} + c$$

Example

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} arc\sin\frac{x}{a} + c$$

52) Prove that $\int \sqrt{(a^2 - x^2)^3} dx = \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2 - x^2} + \frac{3a^4}{8} arc\sin\frac{x}{a} + c$

$$I = \int \sqrt{(a^2 - x^2)^3}\, dx$$

بالتجزيئ-*By Parts*

$u = (a^2 - x^2)^{\frac{3}{2}} \quad , du = \frac{3}{2}(a^2 - x^2)^{\frac{1}{2}} . -2x\, d \quad . dv = dx\,, \quad v = x$

$$I = x\sqrt{(a^2 - x^2)^{\frac{3}{2}}}\, dx + 3\int \underbrace{x^2\sqrt{a^2 - x^2}\, dx}_{55}$$

$$= x\sqrt{(a^2 - x^2)^{\frac{3}{2}}} + \frac{3a^2}{8}\sin^{-1}\left(\frac{x}{a}\right) + \frac{3}{8}x\,(2x^2 - a^2)\sqrt{a^2 - x^2} + c$$

anther Solution

$$I = \int \sqrt{(a^2 - x^2)^3}\, dx$$

$$I = \int \frac{1}{a}\sqrt{\left(1 - \frac{x^2}{a^2}\right)^3}\, dx$$

$let\ \ t = arc\sin\frac{x}{a}$ و $\sin t = \frac{x}{a}$ و $dx = a\cos t\, dt$

$\cos^2 t = 1 - \sin^2 t = 1 - \left(\frac{x}{a}\right)^2 = \frac{a^2 - x^2}{a^2}$ *and* $\sqrt{a^2 - x^2} = a\, cos\, t$

$I = \int (acost)^3 . a\cos t\, dt = a^4 \int \cos^4 t\, dt$

$\cos^4 t = \left(\frac{1+cos2t}{2}\right)^2 = \frac{1}{4}(1 + 2\cos 2t + (cos2t)^2) = \frac{1}{4}(1 + 2\cos 2t + \frac{1+cos4t}{2})$

$= \frac{1}{8}(2 + 4\cos 2t + 1 + cos4t) = \frac{1}{8}(3 + 4\cos 2t + cos4t)$

$I = \frac{a^4}{8}\int (3 + 4cos2t + cos4t)dt = \frac{3a^4}{8}\int dt + \frac{a^4}{8}\int \cos 2t\, dt + \frac{a^4}{5}\int \cos 4\, t\, dt$

$= \frac{3a^4}{8} + \frac{a^4}{4}\sin 2t + \frac{a^4}{32}\sin 4\,t + c$

$\sin 2t = 2\, cost \sin t = 2.\frac{\sqrt{a^2-x^2}}{a} . \frac{x}{a} = \frac{2\,x\sqrt{a^2-x^2}}{a^2}$

$\sin 4\,t = 2\cos 2t . \sin 2t = 2\,(1 - 2\sin^2 t)\sin 2t = 2\left(1 - \frac{2x^2}{a^2}\right). \frac{2x\sqrt{a^2-x^2}}{a^2}$

$= \frac{4}{a^4}(a^2 - 2\,x^2). 2x\sqrt{a^2-x^2} = \frac{4x}{a^4}(a^2 - 2x^2)\sqrt{a^2}$

$I = \frac{3a^4}{8} arc\sin\frac{x}{a} + \frac{a^4}{4}\left(\frac{2x\sqrt{a^2-x^2}}{a^2}\right) + \frac{a^4}{32}\left(\frac{4x}{a^4}(a^2 - 2x^2)\right)$

$= \frac{3a^4}{8}\text{arc}\sin\frac{x}{a} + \frac{a^2x}{2}\sqrt{a^2-x^2} + \frac{x}{a}(a^2 - x^2)\sqrt{a^2-x^2} = \frac{3a^3}{8}\text{arc}\sin\frac{x}{a} + \frac{x}{8}(5\,a^2 - 2x^2)\sqrt{a^2-x^2} + c$

$\int \sqrt{(a^2-x^2)^3}dx = \frac{3a^4}{8} arc\sin\frac{x}{a} + \frac{x}{8}(5a^2 - 2x^2)\sqrt{a^2-x^2} + c$

Example

$$\int \sqrt{(9 - x^2)^3}dx = \frac{x}{8}(5\,9 - 2x^2)\sqrt{9-x^2} + \frac{81}{8} arc\sin\frac{x}{3} + c$$

53)Prove that $\int x\sqrt{a^2-x^2}dx = -\frac{\sqrt{a^2-x^2}}{3} + c$ $\quad$ $let\ a^2 - x^2 = y^2$ و $dx = \frac{-y}{x}dy$

$I = \int x\,y.\frac{-y}{x}dy = \int -y^2 dy = \frac{-y^3}{3} + c = \frac{-\sqrt{(a^2-x^2)^3}}{3} + c$

anther Solution

$u = \sqrt{a^2-x^2}\ , du = \frac{-u}{\sqrt{a^2-x^2}}dx\ , u\,du = -x\,dx$

$\int x\sqrt{a^2-x^2}\,dx = -\int u.u\,du = -\int u^2 du = \frac{-u^3}{3} + c = \int x\sqrt{a^2-x^2}\,dx = \frac{-\sqrt{(a^2-x^2)^3}}{3} + c$

Example

$$\int x\sqrt{25-x^2}dx = -\frac{\sqrt{25-x^2}}{3} + c$$

54)Prove that $\int x\sqrt{(a^2-x^2)^3}\,dx = -\dfrac{\sqrt{(a^2-x^2)^3}}{3} + c$

$let: \; u = \sqrt{a^2-x^2}$ و $du = \dfrac{-u\,dx}{\sqrt{a^2-x^2}}$ و $x\,dx = -u\,du$

$$\int x\sqrt{(a^2-x^2)^3}\,\mathrm{dx} = -\int u^3.u\,du = -\int u^4 du = \frac{-u^5}{5} + c = \frac{-\sqrt{(a^2-x^2)^5}}{5} + c$$

anther Solution

$$I = \int x\,y^{\frac{3}{2}}.\frac{-2dy}{2x} = \frac{-1}{2}\int y^{\frac{3}{2}}\,dy = \frac{-y^{\frac{3}{2}}}{\frac{5}{2}} + c = \frac{-\sqrt{(a^2-x^2)^5}}{5} + c$$

Example

$$\int x\sqrt{(49-x^2)^3}\,dx = -\frac{\sqrt{(49-x^2)^3}}{3} + c$$

55)Prove that $\int x^2\sqrt{a^2-x^2}\,dx = \dfrac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \dfrac{a^2}{8} arc\sin\dfrac{x}{a} + c$

let : $x = a\sin y$ و $dx = a\cos y\,dy$

$$I = \int a^2\sin^2 y\,.a\cos y\sqrt{a^2-a^2\sin^2 y}\;dy = \int a^3\sin^2 y\cos y\sqrt{a^2\cos^2 y\,dy}$$

$$= \int a^4 sin^2 y\cos^2 y\,dy = a^4\int (siny\cos y)^2\;dy \;= a^4\int\left(\frac{1}{2}sin2y\right)^2 dy = \frac{a^4}{4}\sin^2 2y\,dy$$

$$= \frac{a^4}{8}\int(1-\cos 4y)\mathrm{dy} = \frac{a^4}{8}\left(y - \frac{1}{4}\sin 4y\right) + c = \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^4}{16}.2\sin y\cos y\,(1-2\sin^2 y) + c$$

$$= \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) - \frac{a^4}{8}.\frac{\sqrt{a^2-x^2}}{a}.\frac{(-a^2+2x^2)}{a^2} + c = \frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2)} + c$$

anther Solution

بالتجزيئ-*By Parts*

$u = x$ و $du = dx\,, dv = x\sqrt{a^2-x^2}$ و $v = \dfrac{-\sqrt{(a^2-x^2)^3}}{3}$

$$\boldsymbol{I = \frac{-x\sqrt{(a^2-x^2)^3}}{3} + \frac{1}{3}\int\sqrt{(a^2-x^2)^3}dx = \frac{-x\sqrt{(a^2-x^2)^3}}{3} + \frac{1}{3}\left(\frac{x}{8}(a^2-x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}arc\sin\frac{x}{a} + c\right.}$$

$$= \frac{-x\sqrt{(a^2-x^2)^3}}{3} + \frac{x}{24}(a^2-x^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\mathrm{arc}\sin\frac{x}{a} + c$$

$$= \frac{(-8x(a^2-x^2) + x(a^2-2x^2))\sqrt{a^2-x^2}}{24} + \frac{a^4}{8}\mathrm{arc}\sin\frac{x}{a} + c$$

$$= \frac{(-3a^2x + 6x^3)\sqrt{a^2-x^2}}{24} + \frac{a^4 arc\sin\frac{x}{a}}{8} + c = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4 arc\sin\frac{x}{a}}{8} + c$$

Example

$$\int x^2\sqrt{81-x^2}\,dx = \frac{x}{8}(2x^2-81)\sqrt{81-x^2} + \frac{81}{8}\arcsin\frac{x}{9} + c$$

56) Prove that $\int \frac{x^2 dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + c$

let : $x = a\sin y$, $dx = a\cos y\, dy$

$$I = \int \frac{a^3\sin^2 y\cos y}{\sqrt{(a^2-a^2\cos^2 y)^2}}dy = \int \frac{a^3\sin^2 y\cos y}{\sqrt{a^6\cos^6 y}}dy = \int \frac{\sin^2 y\cos y}{\cos^3 y}dy = \int \frac{\sin^2 y}{\cos^2 y}dy = \int \tan^2 y\, dy$$

$$= \int (\sec^2 y - 1)\, dy = \tan y - y + c = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\left(\frac{x}{a}\right) + c$$

anther Solution

$$u = x, du = dx, dv = \frac{x\,dx}{\sqrt{(a^2-x^2)^3}}, v = \frac{1}{\sqrt{a^2-x^2}}$$

$$I = \frac{x}{\sqrt{a^2-x^2}} - \int \frac{dx}{\sqrt{a^2-x^2}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + c$$

Example

$$\int \frac{x^2 dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + c$$

57)Prove that $\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\frac{x}{a+\sqrt{a^2-x^2}} + c$

$x = a\sin t$ و $dx = a\cos t\, dt$

$\sqrt{a^2-x^2} = x\cos t$ و $\cos t = \frac{\sqrt{a^2-x^2}}{x}$

$$\int \frac{a\cos t\, dt}{a\sin t\sqrt{a^2-a^2\sin^2 t}} = \int \frac{\cos t\, dt}{a\sin t\cos t}$$

$$= \frac{1}{a}\int \csc t\, dt$$

$$= \frac{1}{a}\int \frac{\csc t(\cot t + \csc t)}{\cot t + \csc t}dt$$

$$= \frac{-1}{a}\ln(\cot t + \csc t) + c$$

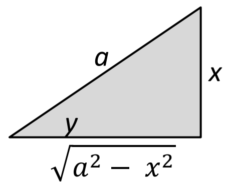
$$= \frac{-1}{a}\ln\left(\frac{\sqrt{a^2-x^2}}{x} + \frac{a}{x}\right) + c = \frac{-1}{a}\ln\left(\frac{\sqrt{a^2-x^2}+a}{x}\right) + c$$

$$= \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2+x^2}}\right) + c$$

57 – *another* Solution

$$\int \frac{dx}{x\sqrt{a^2 - x^2}}$$

let $x = a\sin y$ و $dx = a\cos y\, dy$



$$I = \int \frac{a\cos y\, dy}{a\sin\sqrt{a^2 - a^2\sin^2 y}} = \int \frac{a\cos y\, dy}{\sin y\sqrt{a^2\cos^2 y}} = \int \frac{\cos y}{a\sin y\cos y} dy$$

$$= \int \frac{\sec y}{a} dy = \int \frac{cse\ y\,(csey + \cot y)}{a\,(csey + \cot y)} dy$$

$$= \frac{-1}{a}\ln|\sec y + \cot y| + c = \frac{1}{a}\ln\left|\left(\frac{1}{\sin y} + \frac{\sec y}{\sin y}\right)^{-1}\right| + c = \frac{1}{a}\ln\left|\left(\frac{1 + \sec y}{\sin y}\right)^{-1}\right| + c$$

$$\frac{1}{a}\ln\left|\frac{\sin y}{1 + \sec y}\right| + c = \frac{1}{a}\ln\left|\frac{x}{a\left(1 + \frac{\sqrt{a^2 - x^2}}{a}\right)}\right| + c = \frac{1}{a}\ln\left|\frac{x}{a + \sqrt{a^2 - x^2}}\right| + c$$

another Solution

57) $\int \frac{dx}{x\sqrt{a^2-x^2}}$

on pose $u = \sqrt{a^2-x^2} \Rightarrow x\,dx = -u\,du$ et $x^2 = a^2 - u^2$

$$I = \int \frac{dx}{x\sqrt{a^2-x^2}} = -\int \frac{u\,du}{x^2 \cdot u} = -\int \frac{du}{a^2-u^2} = -\int \frac{du}{(a-u)(a+u)}$$

$$= -\frac{1}{2a}\int\left(\frac{1}{a-u} + \frac{1}{a+u}\right)du = -\frac{1}{2a}\left[-\ln(a-u) + \ln(a+u)\right] + c$$

$$= \frac{1}{2a}\ln\frac{a-u}{a+u} + c = \frac{1}{2a}\ln\frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + C$$

$$= \frac{1}{2a}\ln\frac{\left(a-\sqrt{a^2-x^2}\right)^2}{a^2-(a^2-x^2)} + C = \frac{1}{2a}\ln\frac{\left(a-\sqrt{a^2-x^2}\right)^2}{x^2} + C$$

$$= \frac{1}{a}\ln\frac{a-\sqrt{a^2-x^2}}{x} + C = \frac{1}{a}\ln\frac{\left(a-\sqrt{a^2-x^2}\right)\left(a+\sqrt{a^2-x^2}\right)}{x\left(a+\sqrt{a^2-x^2}\right)} + C$$

$$= \frac{1}{a}\ln\frac{x^2}{x\left(a+\sqrt{a^2-x^2}\right)} + C = \frac{1}{a}\ln\frac{x}{a+\sqrt{a^2-x^2}}$$

$$\boxed{\int\frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln\frac{x}{a+\sqrt{a^2-x^2}} + C}$$